ANOTHER PROOF OF UNIQUENESS OF SOLUTIONS OF RICCI FLOW ON COMPLETE NONCOMPACT MANIFOLDS

Shu-Yu Hsu

Department of Mathematics
National Chung Cheng University
168 University Road, Min-Hsiung
Chia-Yi 621, Taiwan, R.O.C.
e-mail:syhsu@math.ccu.edu.tw

Aug 10, 2010

Abstract. We give a simple proof of the uniqueness of solutions of the Ricci flow on complete noncompact manifolds with bounded curvatures using the De Turck approach. As a consequence we obtain a correct proof of the existence of solution of the Ricci harmonic flow on complete noncomplete manifolds with bounded curvatures.

Recently there is a lot of study on the Ricci flow on manifolds by R. Hamilton [H1–3] and others. Existence of solution $(M, g(t))$, $0 \le t \le T$, of the Ricci flow equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{0.1}$$

on compact manifold M where $R_{ij}(t)$ is the Ricci curvature of $g(t)$ and $g_{ij}(x, 0) = g_{ij}(x)$ is a smooth metric on M is proved by R. Hamilton in [H1]. R. Hamilton [H1] also proved that when $g_{ij}(x)$ is a metric of strictly positive Ricci curvature, then the evolving metric will converge modulo scaling to a metric of constant positive curvature.

Since the proof of existence of solution of the Ricci flow in [H1] is very hard, later D.M. DeTurck [D] deviced another method to prove existence and uniqueness of solution of Ricci flow on compact manifolds. Let M be a n-dimensional manifold with $(M, g_{ij}(t))$, $0 \le t \le T$, being a solution of the Ricci flow (0.1) and let $(N, h_{\alpha\beta})$ be a fixed n-dimensional manifold. He introduced the associated Ricci harmonic flow $F = (F^\alpha) : (M, g(t)) \to (N, h)$ given by

$$\frac{\partial F}{\partial t} = \Delta_{g(t),h} F \tag{0.2}$$

1991 *Mathematics Subject Classification.* Primary 58J35, 53C43 Secondary 35K55.

Key words and phrases. Ricci harmonic flow, Ricci flow, existence, uniqueness.

where

$$\Delta_{g(t),h} F^\alpha = \Delta_{g(t)} F^\alpha + g^{ij}(x,t)\widetilde{\Gamma}^\alpha_{\beta,\gamma}(F(x,t))\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j} \tag{0.3}$$

in the local co-ordinates $x=(x^1,\dots,x^n)$ of the domain manifold $(M, g_{ij}(t))$ and the local co-ordinates (y^α) of the target manifold $(N, h_{\alpha\beta})$ with

$$\Delta_{g(t)} F^\alpha = g^{ij}\nabla_i\nabla_j F^\alpha$$

and $\widetilde{\Gamma}^\alpha_{\beta,\gamma}$ being the Christoffel symbols of $(N, h_{\alpha\beta})$. When the solution $F(\cdot,t)$ of (0.2) is a diffeomorphism, this harmonic map flow induces a push forward metric

$$\hat{g}(t) = (F)_*(g(t)) = (F(\cdot,t)^{-1})^*(g(t)) \tag{0.4}$$

on the target manifold N which satisfies the Ricci-DeTurck flow [H3],

$$\frac{\partial}{\partial t}\hat{g}_{\alpha\beta} = (L_V\hat{g})_{\alpha\beta} - 2\hat{R}_{\alpha\beta} \tag{0.5}$$

for some time varying vector field V on the target manifold N where $\hat{R}_{\alpha\beta}$ is the Ricci curvature associated with the metric $\hat{g}(t)$. Since (0.5) is strictly parabolic [H3], it is easiler to solve (0.5) than (0.1) which is weakly parabolic [H1]. The existence and uniqueness of solutions of Ricci flow on compact manifolds are then reduced to the study of existence and other properties of the harmonic map flow (0.2) and the Ricci-DeTurck flow (0.5). We refer the reader to the paper [H3] of R. Hamilton on a sketch of this approach on compact manifolds.

Naturally one would expect this approach should also work for non-compact complete Riemmanian manifolds. In [S1] W.X. Shi used this technique to prove the existence of solution of (0.1) on complete non-compact Riemannian manifolds. In [LT] P. Lu and G. Tian used the De Turck trick to prove the uniqueness of the standard solution of Ricci flow on $\mathbb{R}^n$, $n\ge 3$, which is radially symmetric about the origin. Recently S.Y. Hsu [Hs] extended the result of [LT] and proved the uniqueness of the solution of the radially symmetric solution of the Ricci harmonic flow (0.2) assoicated with the standard solution of Ricci flow.

In [CZ] B.L. Chen and X.P. Zhu attempted to prove the uniqueness of solutions of the Ricci flow on complete non-compact manifolds by using the De Turck trick. However their proof is not correct because the crucial lemma Lemma 2.2 of [CZ] is not correct. In Lemma 2.2 of [CZ] they claimed that they can construct a sequence of functions $\{\phi_a\}_{a\ge 1}$ which behaves like the distance function and have bounded covariant derivatives of all orders. They do this by smoothing the distance function with the Riemannian convolution operator of R.E. Green and H. Wu (P.646–647 of [GW1] and P.57 of [GW2]). More precisely ([GW1],[GW2]) let $\psi:\mathbb{R}\to\mathbb{R}$ be a nonnegative C^∞ function with support in $[-1,1]$ which is constant in a neighborhood of 0 and

$$\int_{v\in\mathbb{R}^n}\psi(|v|) = 1.$$

Then for any n-dimensional complete non-compact Riemannian manifold (M,g) and continuous function $f: M \to \mathbb{R}$, the Riemannian convolution operator for f is defined as

$$f_\varepsilon(p) = \frac{1}{\varepsilon^n} \int_{v \in T_pM} f(exp_p v)\psi(|v|/\varepsilon)\, d\Omega_p \quad \forall \varepsilon > 0, p \in M$$

where $d\Omega_p$ is the measure on T_pM obtained from the Riemannian metric on M and $exp_p : T_pM \to M$ is the exponential map of M at p. Hence

$$f_\varepsilon(p) = \frac{1}{\varepsilon^n} \int_{q \in M} f(q)\psi(|exp_p^{-1}(q)|/\varepsilon)\, dq \quad \forall 0 < \varepsilon < \mathrm{inj}(p), p \in M \tag{0.6}$$

As observed by R.E. Green and H. Wu (P.646–647 of [GW1]) for the smoothness of f_ε at $p \in M$, 2ε has to be less than the injectivity radius $\mathrm{inj}(p)$ of M at p. This is because one has to use the representation (0.6) for f_ε in order to pass all the derivatives of f_ε onto the smooth function

$$\psi(|exp_p^{-1}(q)|/\varepsilon)$$

in the integrand in (0.6). Thus one needs $exp_p : T_pM \to M$ to be a local diffeomorphism between the ball $B(0,2\varepsilon)$ in T_pM and the ball $B_g(p,2\varepsilon) \subset M$ for $0 < 2\varepsilon < \mathrm{inj}(p)$, $p \in M$, and

$$|\nabla^k f_\varepsilon|(p) \approx C\varepsilon^{-k} \ge C\,\mathrm{inj}(p)^{-k} \quad \forall 0 < 2\varepsilon < \mathrm{inj}(p), p \in M, k \in \mathbb{Z}^+. \tag{0.7}$$

Let $p_0 \in M$ be a fixed point of M and suppose that M also has bounded curvature. Since the injectivity radius $\mathrm{inj}(p)$ may decreases to 0 as $dist(p_0,p) \to \infty$ [CGT], [CLY], by (0.7) $|\nabla^k f_\varepsilon|(p)$ is not uniformly bounded on M in general for any $k \in \mathbb{Z}^+$. Thus the sequence of functions $\{\phi_a\}_{a\ge 1}$ constructed in [CZ] can behave like

$$|\nabla^k \phi_a|(p) \approx C\,\mathrm{inj}(p)^{-k} \quad \forall p \in M, k \in \mathbb{Z}^+$$

and tends to infinity as $dist(p_0,p) \to \infty$. Hence Lemma 2.2 of [CZ] is not correct.

A simple example of sequence of manifolds with bounded curvature and injectivity radii tending to zero is as follows. Consider the manifolds $M = S^1 \times \mathbb{R}$ with the product metric $g_\delta = (\delta \cdot h) \times dx$, $\delta > 0$, where h is the standard metric on the unit circle S^1 (i.e. the metric induced on S^1 by wrapping $\mathbb{R}$ onto S^1) and dx is the Euclidean metric on the real line $\mathbb{R}$. Then the manifold (M, g_δ) is flat with curvature $\mathrm{Rm}(g_\delta) \equiv 0$ on M but the injectivity radius $i_\delta(p)$, $p \in M$, of (M, g_δ) is constant on M and is equal to the conjugate radius $\mathrm{conj}_\delta(p)$ at p for any $p \in M$. Moreover we have $\mathrm{conj}_\delta(p) = i_\delta(p) = \pi\delta \to 0$ as $\delta \to 0$ for any $p \in M$.

Similarly according to the results and examples in [CG] there are many examples of sequences of manifolds with uniformly bounded curvature but with the corresponding injectivity radii converging uniformly to zero. One can also read the survey article [G] by J.D.E. Grant on the injectivity radius estimate, the paper by J. Cheeger, M. Gromov and M. Taylor [CGT], and (i) of Remark 1.7 of [AM] for various lower bound estimates on a manifold under various curvature conditions. Hence Lemma 2.2 of [CZ] cannot be correct.

Observe that the proof of [CZ] uses the uniform boundedness property of the higher order covariant derivatives of the approximate distance function of Lemma 2.2 of [CZ] in an essential way. In this paper instead of Lemma 2.2 of [CZ] I will use Corollary 1.5 of this paper in the Deturck program to solve the uniqueness problem. Because of the absence of the uniform boundedness property of third and higher order derivatives of the approximate distance function in Corollary 1.5, many theorems in this paper require new proofs different from that of [CZ].

In the book [SY] by R. Schoen and S.T. Yau a weaker result similar to Corollary 1.5 is proved in Theorem 4.2 of Chapter 1 using P.D.E. methods. However my proof of Corollary 1.5 is more elementary and requires only knowledge of the distance function on the manifold.

Note that the manifold under consideration is non-compact and Lemma 3.5 of [H2] is applicable to the proof of the inequality on the last two lines of P.144 of [CZ] only when M is compact or the extremum of the norm of the covariant derivatives of the solution of Ricci harmonic map (0.2) can be attained in a compact set of M independent of time. Hence the proof of the uniform estimates over M for the norms of the covariant derivatives of solutions of the harmonic map and the uniform lower bound estimate for the existence time of the solutions of the approximate problems in Theorem 2.6 of [CZ] on P.144–145 of [CZ] is also not correct. Thus the proof of the existence of the Ricci harmonic map in Theorem 2.1 of [CZ] is not correct.

The proof of Proposition 3.1 of [CZ] has gaps and the proof of Proposition 3.3 of [CZ] which is crucial to the proof of uniqueness of solutions of Ricci flow is also not correct since the deduction of the last two inequalities on P.151 for the proof of Proposition 3.3 of [CZ] assumed that one can interchange differentiation and taking limit as $\varepsilon \to 0$ which is also not true in general.

In this paper we will give a correct proof of the uniqueness of solutions of the Ricci flow on complete noncompact manifolds with bounded curvatures. We will use the De Turck approach to prove this result. We will prove the existence of solution of the Ricci harmonic flow on complete noncompact manifolds with bounded curvatures.

The plan of the paper is as follows. In section 1 we will prove various estimates for the Hessian of the distance functions in both (M,g) and the target manifold (N,h). We will construct C^2 functions on M with uniformly bounded first and second order covariant derivatives which approximate the distance function of $(M,g(0))$. In section 2 we will construct solutions of (0.2) in bounded cylindrical domains with Dirichlet boundary condition and in $M\times(0,T_1)$ for some constant $T_1>0$. We will prove the uniform estimates on the norm of the covariant derivatives of the solutions of the Ricci harmonic flow. In section 3 we will prove the uniqueness of the solutions of Ricci flow on complete noncompact manifolds with bounded curvatures.

We will let $(M,g(t))$, $0\le t\le T$, be a solution of the Ricci flow on a n-dimensional complete non-compact manifold and $(N,h_{\alpha\beta})=(M,g(0))$ for the rest of the paper. We will assume that there exists a constant $k_0>0$ such that

$$|\mathrm{Rm}|\le k_0 \quad \text{on } M\times[0,T] \tag{0.8}$$

where Rm is the Riemannian curvature of $g(t)$ and $|\cdot|$ is the norm with respect to the

metric $g(t)$. Note that by the results of W.X. Shi [S1] for any $m \in \mathbb{Z}^+$ there exists a constant $c_m > 0$ such that

$$|\nabla^m R_{ijkl}| \le c_m t^{-\frac{m}{2}}. \tag{0.9}$$

For any $p, q \in M$, we let $\rho(p, q)$ be the distance between p and q with respect to $g(0)$. For any $k > 0$, $p \in M$, let $B(p, k) = \{q \in M : \rho(p, q) < k\}$. We will fix a point $p_0 \in M$. For any $k > 0$, let $B_k = B(p_0, k)$ and $Q_k^{T_1} = B(p_0, k) \times (0, T_1)$ for any $T_1 > 0$. For any bounded domain $\Omega \subset M$ we let

$$\partial_p(\Omega \times (0, T_1)) = \overline{\Omega} \times \{0\} \cup \partial\Omega \times [0, T_1)$$

be the parabolic boundary of $\Omega \times (0, T_1)$. For any open set $O \subset M$, we let $\mathrm{Vol}_{g(t)}(O)$ be the volume of O with respect to the metric $g(t)$. For any $r > 0$, let $V_{-k_0}(r)$ be the volume of a geodesic ball of radius r in a space form of curvature $-k_0$. For any point $x \in M$ we let $\mathrm{Cut}(x)$ be the set of all cut points of x with respect to the metrics $g(0)$.

Let $\nabla, \nabla^0, \widetilde{\nabla}, \hat{\nabla}$ be the covariant derivatives with respect the the metric $g(t)$, $g(0)$, h, and $\hat{g}$ respectively. Let $\Gamma_{ij}^k(t)$, $\widetilde{\Gamma}_{ij}^k(t)$, R_{ijkl}, $\widetilde{R}_{ijkl}$, R_{ij}, $\widetilde{R}_{ij}$, be Christoffel symbols, curvature tensors and Ricci tensors with respect to the metric $g(t)$ and h respectively. Let Δ_t be the Laplace operator with respect to the metric $g(t)$.

We let $f : (M, g(0)) \to (N, h)$ be a given diffeomorphism satisfying

$$K_1 = \sup_M |\nabla f|_{g(0),h} = \sup_M \left(g^{ij}(0) h_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} \right)^{\frac{1}{2}} < \infty \tag{0.10}$$

and

$$K_2 = \sup_M |\nabla^2 f|_{g(0),h} < \infty. \tag{0.11}$$

for the rest of the paper. When there is no ambiguity we will drop the subscript and write $|\nabla f|$, $|\nabla^2 f|$, for $|\nabla f|_{g(0),h}$ and $|\nabla^2 f|_{g(0),h}$.

By the discussion in [J] (cf. [H3]) we can write the derivative ∇F as

$$\nabla F = \frac{\partial F^\alpha}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^\alpha}$$

and consider ∇F as a section of the bundle $T^*M \otimes F^{-1}TN$. The connection ∇ on $(M, g(t))$ and the connection $\widetilde{\nabla}$ on (N, h) induce a natural connection ∇ on $T^*M \otimes F^{-1}TN$ which can in turn be extended naturally to a connection on $T^*M \otimes T^*M \otimes F^{-1}TN$. By induction we get a natural connection on $T^*M^{\otimes p} \otimes F^{-1}TN$ for any $p \in \mathbb{Z}^+$ from the connection ∇ on $(M, g(t))$ and the connection $\widetilde{\nabla}$ on (N, h). More precisely for any (cf. [J], [H3], [CZ])

$$u = u^\alpha_{i_1, i_2, \dots, i_{p-1}} dx_{i_1} \otimes \cdots dx_{i_{p-1}} \otimes \frac{\partial}{\partial y^\alpha} \in T^*M^{\otimes(p-1)} \otimes F^{-1}TN, p \ge 2, \tag{0.12}$$

$$\nabla u = (\nabla_{i_p} u^\alpha_{i_1, i_2, \dots, i_{p-1}})\, dx_{i_1} \otimes \cdots dx_{i_p} \otimes \frac{\partial}{\partial y^\alpha} \in T^*M^{\otimes p} \otimes F^{-1}TN$$

where

$$\nabla_{i_p} u^{\alpha}_{i_1,i_2,\dots,i_{p-1}} = \frac{\partial}{\partial x^{i_p}} u^{\alpha}_{i_1,i_2,\dots,i_{p-1}} - \Gamma^{m}_{i_p,i_j} u^{\alpha}_{i_1,\dots,i_{j-1},m,i_j,\dots,i_{p-1}} + \widetilde{\Gamma}^{\alpha}_{\beta\gamma} \frac{\partial F^{\beta}}{\partial x^{i_p}} u^{\gamma}_{i_1,\dots,,i_{p-1}}.$$

For the interchange of two covariant derivatives on $T^*M^{\otimes p} \otimes F^{-1}TN$ we have (cf. P.258 of [H1] and P.133 of [CZ])

$$\nabla_i \nabla_j u^{\alpha}_{i_1,\dots,i_p} - \nabla_j \nabla_i u^{\alpha}_{i_1,\dots,i_p} = R_{iji_m l} g^{lm} u^{\alpha}_{i_1,\dots,m,\dots,i_p} + \widetilde{R}_{pqrs} \frac{\partial F^p}{\partial x^i} \frac{\partial F^q}{\partial x^j} h^{\alpha r} u^{s}_{i_1,\dots,i_p}.$$

We will equip $T^*M^{\otimes p} \otimes F^{-1}TN$ with the norm $g^{\otimes p} \otimes h$ induced from g and h. We write $|u|_{g(t),h}$ for the norm of $u \in T^*M^{\otimes p} \otimes F^{-1}TN$. When there is no ambiguity we will drop the subscript and write $|u|$ instead of $|u|_{g(t),h}$. By abuse of notation we will also denote the operator $g^{ij}\nabla_i\nabla_j$ on $u^{\alpha}_{i_1,\dots,i_p}$ by Δ_t.

Similarly (cf. P.133 of [CZ]) the time derivative $\partial/\partial t$ can be extended naturally to a covariant time derivative ∇_t on $T^*M^{\otimes p} \otimes F^{-1}TN$ for any $p \geq 1$. For any $u \in T^*M^{\otimes p} \otimes F^{-1}TN$, $p \geq 1$ given by (0.12), we define

$$\nabla_t u = (\nabla_t u^{\alpha}_{i_1,i_2,\dots,i_p})\, dx_{i_1} \otimes \cdots dx_{i_p} \otimes \frac{\partial}{\partial y^{\alpha}}$$

where

$$\nabla_t u^{\alpha}_{i_1,i_2,\dots,i_p} = \frac{\partial}{\partial t} u^{\alpha}_{i_1,i_2,\dots,i_p} + u^{\gamma}_{i_1,\dots,,i_p} \widetilde{\Gamma}^{\alpha}_{\beta\gamma} \frac{\partial F^{\beta}}{\partial t}.$$

Let $\eta \in C^{\infty}(\mathbb{R})$, $0 \leq \eta \leq 1$, be an even function such that $\eta(x) \equiv 1$ for $|x| \leq 1/2$, $\eta(x) \equiv 0$ for $|x| \geq 1$, and $\eta'(x) \leq 0$ for $x \geq 0$. Let $\phi = \eta^2$. Then $\phi \in C^{\infty}(\mathbb{R})$ is an even function, $0 \leq \phi \leq 1$, $\phi(x) \equiv 1$ for $|x| \leq 1/2$, $\phi(x) \equiv 0$ for $|x| \geq 1$, $\phi'(x) \leq 0$ for $x \geq 0$, and

$$\sup_{\mathbb{R}} ({\phi'}^2/\phi) = 4 \sup_{\mathbb{R}} {\eta'}^2 < \infty. \tag{0.13}$$

Section 1

In this section we will use a modification of the method of W.X. Shi [S2] to construct C^2 functions on M with uniformly bounded first and second order covariant derivatives which approximate the distance function of $(M, g(0))$. We will obtain a sequence of regularization $\{\overset{a}{h}\}$ for the metric h each of which has a uniform lower bound on the injectivity radius on N. We first prove some estimates for the Hessian of the distance functions on M.

Lemma 1.1. *Let $y_1, y_2 \in M$ with $y_2 \notin Cut(y_1)$ and*

$$\rho(y_1, y_2) \leq \pi/4\sqrt{k_0} \tag{1.1}$$

Let γ be the unique minimal geodesic in $(M, g(0))$ from y_1 to y_2. Then for any unit vector $X \in T_{y_2}M$ perpendicular to $\partial/\partial\gamma$,

$$Hess_{g(0)}(\rho)(X, X) \geq \frac{\pi}{4\rho(y_1, y_2)} \tag{1.2}$$

where $\rho = \rho(y_1, y_2)$. Hence for any unit vector $X \in T_{y_2}M$,

$$\begin{cases} Hess_{g(0)}(\rho^2/2)(X,X) \ge \dfrac{\pi}{4} \\ Hess_{g(0)}(\rho)(X,X) \ge 0. \end{cases} \tag{1.3}$$

Proof. Let N_1 be a space form of curvature k_0. Let $\rho_{N_1}(z)$ be the distance function on N_1 with respect to some fixed point $z_1 \in N_1$. Suppose z_2 is a point on N_1 such that $\rho(y_1, y_2) = \rho_{N_1}(z_2)$. Let $\rho = \rho(y_1, y_2)$ and let γ_1 be the minimal geodesics in N_1 from z_1 to z_2. Let $\xi \in T_{z_2}N_1$ be a unit vector which satisfies $<\xi, \partial/\partial\gamma_1>= 0$. We extend ξ to a vector field X_1 perpendicular to $\partial/\partial\gamma_1$ along γ_1 by parallel translation. Let

$$f(s) = \frac{\sin\sqrt{k_0}s}{\sin\sqrt{k_0}\rho}.$$

Then by (1.1) and an argument similar to that of [SY], $Y(s) = f(s)X_1(s)$ is the Jacobi field with $Y(\rho_0) = \xi$ and

$$\begin{aligned} Hess_{g(0)}(\rho_{N_1})(\xi,\xi)(z) =& \int_0^\rho \left(\left|\frac{df}{ds}\right|^2 - k_0 f(s)^2 \right) ds \\ =& \frac{k_0}{\sin^2\sqrt{k_0}\rho} \int_0^\rho (\cos^2\sqrt{k_0}s - \sin^2\sqrt{k_0}s)\, ds \\ =& \frac{\sqrt{k_0}\sin 2\sqrt{k_0}\rho}{2\sin^2\sqrt{k_0}\rho} \\ =& \frac{1}{\rho}\cdot\frac{\sqrt{k_0}\rho}{\tan\sqrt{k_0}\rho} \\ \ge& \frac{1}{\rho}\cdot\frac{\pi/4}{\tan(\pi/4)} \\ \ge& \frac{\pi}{4\rho(y_1, y_2)}. \end{aligned} \tag{1.4}$$

Hence by (0.8), (1.4), and the Hessian comparsion theorem [SY], for any unit vector $X \in T_{y_2}M$ satisfying $<X, \partial/\partial\gamma>= 0$, we have

$$\begin{aligned} & Hess_{g(0)}(\rho)(X,X) \ge Hess_{g(0)}(\rho_{N_1})(\xi,\xi) \ge \frac{\pi}{4\rho(y_1,y_2)} \\ \Rightarrow\quad & Hess_{g(0)}(\rho^2/2)(X,X) \ge \frac{\pi}{4} \end{aligned} \tag{1.5}$$

and (1.2) follows. By direct computation, for any $X \in T_{y_2}M$ satisfying $<X, \partial/\partial\gamma>= 0$,

$$\begin{cases} Hess_{g(0)}(\rho)(\partial/\partial\gamma, \partial/\partial\gamma) = Hess_{g(0)}(\rho)(X, \partial/\partial\gamma) = Hess_{g(0)}(\rho)(\partial/\partial\gamma, X) = 0 \\ Hess_{g(0)}(\rho^2/2)(X, \partial/\partial\gamma) = Hess_{g(0)}(\rho^2/2)(\partial/\partial\gamma, X) = 0 \\ Hess_{g(0)}(\rho^2/2)(\partial/\partial\gamma, \partial/\partial\gamma) = 1. \end{cases} \tag{1.6}$$

For any $X \in T_{y_2}M$,

$$X = X_1 + \lambda \frac{\partial}{\partial \gamma}$$

for some constant λ and $X_1 \in T_{y_2}M$ perpendicular to $\partial/\partial\gamma$. Then by (1.5) and (1.6), $\forall X \in T_{y_2}M$,

$$\begin{aligned}
Hess_{g(0)}(\rho)(X,X) =& Hess_{g(0)}(\rho)(X_1,X_1) + 2\lambda Hess_{g(0)}(\rho)(X, \frac{\partial}{\partial\gamma}) \\
&+ \lambda^2 Hess_{g(0)}(\rho)(\frac{\partial}{\partial\gamma}, \frac{\partial}{\partial\gamma}) \\
\geq& 0
\end{aligned}$$

and

$$\begin{aligned}
Hess_{g(0)}(\rho^2/2)(X,X) =& Hess_{g(0)}(\rho^2/2)(X_1,X_1) + 2\lambda Hess_{g(0)}(\rho^2/2)(X, \frac{\partial}{\partial\gamma}) \\
&+ \lambda^2 Hess_{g(0)}(\rho^2/2)(\frac{\partial}{\partial\gamma}, \frac{\partial}{\partial\gamma}) \\
\geq& \frac{\pi}{4}|X_1|^2_{g(0)} + \lambda^2 \\
\geq& \frac{\pi}{4}
\end{aligned}$$

and (1.3) follows.

Lemma 1.2. *(cf. P.225-226 of* [S1]*) There exist constants $c_2 > c_1 > 0$ such that*

$$\begin{cases} c_1 g_{ij}(x,0) \leq g_{ij}(x,t) \leq c_2 g_{ij}(x,0) & \forall 0 \leq t \leq T \\ c_1 g^{ij}(x,0) \leq g^{ij}(x,t) \leq c_2 g^{ij}(x,0) & \forall 0 \leq t \leq T. \end{cases} \tag{1.7}$$

Lemma 1.3. *Let $x_0, x \in B_k$ with $x \notin Cut(x_0)$ and let $\rho(x) = \rho(x_0,x)$ satisfy (1.1). Then exists a constant $C_1 > 0$ such that*

$$-C_1\sqrt{t} \leq \Delta_t \rho \leq C_1(1 + \frac{1}{\rho}) \quad \forall 0 \leq t \leq T. \tag{1.8}$$

Proof. Note that

$$\Delta_t \rho = g^{ij}(t)\nabla_i\nabla_j\rho = g^{ij}(t)\biggl(\nabla_i^0\nabla_j^0\rho - (\Gamma_{ij}^k(t) - \Gamma_{ij}^k(0))\frac{\partial\rho}{\partial x^k}\biggr). \tag{1.9}$$

We choose a normal coordinate system $\{\partial/\partial x^i\}$ with respect to the metric $g_{ij}(t)$ at x. Then by (0.1), (0.8), (0.9), Lemma 1.2, and an argument similar to the proof of (2.16) and (2.17) of [CZ],

$$\left|\frac{\partial}{\partial t}\Gamma_{ij}^k(t)\right| \leq \frac{C}{\sqrt{t}} \quad \Rightarrow \quad |\Gamma_{ij}^k(t) - \Gamma_{ij}^k(0)| \leq C\sqrt{t} \quad \forall 0 \leq t \leq T. \tag{1.10}$$

By the Hessian comparison theorem [SY], P.309-310 of [S2], and Lemma 1.2 (cf. P.136 of [CZ]),

$$\nabla_i^0\nabla_j^0\rho \le \frac{1+\sqrt{k_0}\rho}{\rho}g_{ij}(x,0)$$
$$\Rightarrow \quad g^{ij}(x,t)\nabla_i^0\nabla_j^0\rho \le c_2\frac{1+\sqrt{k_0}\rho}{\rho}g^{ij}(x,0)g_{ij}(x,0) \le nc_2\left(\frac{1+\sqrt{k_0}\rho}{\rho}\right). \tag{1.11}$$

By Lemma 1.1 (1.3) holds. Hence

$$g^{ij}(x,t)\nabla_i^0\nabla_j^0\rho = \nabla_i^0\nabla_i^0\rho \ge 0. \tag{1.12}$$

By (1.9), (1.10), (1.11) and (1.12) we get (1.8) and the lemma follows.

Let

$$k_1 = \pi/4\sqrt{k_0}$$

and

$$\overline{\rho}(y) = \overline{\rho}(p_0,y) = \frac{\int_M \rho(p_0,z)\eta(\rho(y,z)/k_1)\,dz}{\int_M \eta(\rho(y,z)/k_1)\,dz}.$$

Then

$$\rho(p_0,y) - k_1 \le \overline{\rho}(y) \le \rho(p_0,y) + k_1 \quad \forall y \in M. \tag{1.13}$$

Lemma 1.4. *$\overline{\rho} \in C^2(M)$ and there exists a constant $C_1 >$ such that*

$$\begin{cases} |\widetilde{\nabla}\overline{\rho}(y)|_h \le C_1 & \forall y \in M \\ |\widetilde{\nabla}^2\overline{\rho}(y)|_h \le C_1 & \forall y \in M \end{cases}$$

Proof. Since

$$z \in M \setminus \mathrm{Cut}\,(y) \quad \Leftrightarrow \quad y \in M \setminus \mathrm{Cut}\,(z), \tag{1.14}$$

$$|\widetilde{\nabla}\rho(y,z)|_h \le 1 \quad \forall z \in M \setminus \mathrm{Cut}\,(y). \tag{1.15}$$

As Cut(y) has measure zero, by (1.15) and the Lebesgue dominated convergence theorem,

$$\widetilde{\nabla}_\beta\overline{\rho}(y) = k_1^{-1}\frac{\int_M (\rho(p_0,z)-\overline{\rho}(y))\eta'\widetilde{\nabla}_\beta\rho(y,z)\,dz}{\int_M \eta(\rho(y,z)/k_1)\,dz} \tag{1.16}$$

where $\eta' = \eta'(\rho(y,z)/k_1)$. By (1.13), (1.15), (1.16) and the volume comparison theorem [SY], [Ch], there exist constants $C_1 > 0$ and $C_1' > 0$ such that

$$|\widetilde{\nabla}\overline{\rho}(y)|_h \le (C_1'/k_1)\frac{\mathrm{Vol}_h(B_h(p_0,k_1))}{\mathrm{Vol}_h(B_h(p_0,k_1/2))}\cdot \sup_{z\in B_h(y,k_1)}|\rho(p_0,z)-\overline{\rho}(y)| \le 2C_1'\frac{V_{-k_0}(k_1)}{V_{-k_0}(k_1/2)} \le C_1 \tag{1.17}$$

for any $y \in M$. By the Hessian comparison theorem [SY], Lemma 1.1 (cf. P.309–311 of [S2]), and (1.14),

$$\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \rho(y,z) \le \frac{1+\sqrt{k_0}\rho}{\rho} h_{\alpha\beta} \quad \forall z \in M \setminus \mathrm{Cut}(y) \tag{1.18}$$

and

$$\begin{aligned}
& Hess_{g(0)}(\rho(y,z))(\frac{\partial}{\partial y^\alpha}+\frac{\partial}{\partial y^\beta},\frac{\partial}{\partial y^\alpha}+\frac{\partial}{\partial y^\beta}) \ge 0 \\
\Rightarrow \quad & Hess_{g(0)}(\rho(y,z))(\frac{\partial}{\partial y^\alpha},\frac{\partial}{\partial y^\beta}) \\
& \ge -\frac{1}{2}\left(Hess_{g(0)}(\rho(y,z))(\frac{\partial}{\partial y^\alpha},\frac{\partial}{\partial y^\alpha})+Hess_{g(0)}(\rho(y,z))(\frac{\partial}{\partial y^\beta},\frac{\partial}{\partial y^\beta})\right)
\end{aligned} \tag{1.19}$$

holds for any $z \in M \setminus \mathrm{Cut}\,(y)$. We now choose a normal co-ordinate system $\{\partial/\partial y^\alpha\}$ at y. Then by (1.15), (1.18), and (1.19),

$$|\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \rho|_h \le n^2\left(\frac{1+\sqrt{k_0}\rho}{\rho}\right) \quad \forall z \in M \setminus \mathrm{Cut}(y) \tag{1.20}$$

$$\Rightarrow \quad |\rho(p_0,z)(\eta'' \widetilde{\nabla}_\alpha \rho \widetilde{\nabla}_\beta \rho + k_1 \eta' \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \rho)|_h \le C_2(\rho(p_0,y)+1)\chi_{B_h(y,k_1)} \tag{1.21}$$

holds for any $z \in M \setminus \mathrm{Cut}(y)$ where $\eta''=\eta''(\rho(y,z)/k_1)$ and $\chi_{B_h(y,k_1)}$ is the characteristic function of the set $B_h(y,k_1)$. Hence by (1.16), (1.21), and the Lebesgue dominated

convergence theorem,

$$
\begin{aligned}
\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\overline{\rho}(y) =& k_1^{-2}\Bigg\{\frac{\int_M \rho(p_0,z)(\eta''\widetilde{\nabla}_\alpha\rho\widetilde{\nabla}_\beta\rho + k_1\eta'\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\rho)\,dz}{\int_M \eta(\rho(y,z)/k_1)\,dz}\\
&- \frac{\int_M \rho(p_0,z)\eta'\widetilde{\nabla}_\beta\rho\,dz}{\int_M \eta(\rho(y,z)/k_1)\,dz}\cdot\frac{\int_M \eta'\widetilde{\nabla}_\alpha\rho\,dz}{\int_M \eta(\rho(y,z)/k_1)\,dz}\\
&- k_1\widetilde{\nabla}_\alpha\overline{\rho}(y)\frac{\int_M \eta'\widetilde{\nabla}_\beta\rho\,dz}{\int_M \eta(\rho(y,z)/k_1)\,dz}\\
&- \overline{\rho}(y)\frac{\int_M (\eta''\widetilde{\nabla}_\alpha\rho\widetilde{\nabla}_\beta\rho + k_1\eta'\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\rho)\,dz}{\int_M \eta(\rho(y,z)/k_1)\,dz}\\
&- \overline{\rho}(y)\frac{\int_M \eta'\widetilde{\nabla}_\alpha\rho\,dz}{\int_M \eta(\rho(y,z)/k_1)\,dz}\cdot\frac{\int_M \eta'\widetilde{\nabla}_\beta\rho\,dz}{\int_M \eta(\rho(y,z)/k_1)\,dz}\Bigg\}\\
=& k_1^{-2}\Bigg\{\frac{\int_M (\rho(p_0,z)-\overline{\rho}(y))(\eta''\widetilde{\nabla}_\alpha\rho\widetilde{\nabla}_\beta\rho + k_1\eta'\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\rho)\,dz}{\int_M \eta(\rho(y,z)/k_1)\,dz}\\
&- \frac{\int_M (\rho(p_0,z)-\overline{\rho}(y))\eta'\widetilde{\nabla}_\beta\rho\,dz}{\int_M \eta(\rho(y,z)/k_1)\,dz}\cdot\frac{\int_M \eta'\widetilde{\nabla}_\alpha\rho\,dz}{\int_M \eta(\rho(y,z)/k_1)\,dz}\\
&- k_1\widetilde{\nabla}_\alpha\overline{\rho}(y)\frac{\int_M \eta'\widetilde{\nabla}_\beta\rho\,dz}{\int_M \eta(\rho(y,z)/k_1)\,dz}\Bigg\}
\end{aligned}
\tag{1.22}
$$

where $\rho=\rho(y,z)$. By (1.13), (1.17), (1.20), (1.21), (1.22), and the volume comparison theorem,

$$
\begin{aligned}
&|\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\overline{\rho}(y)|_h\\
\le& C_2'\left\{\frac{\mathrm{Vol}_h(B_h(p_0,k_1))}{\mathrm{Vol}_h(B_h(p_0,k_1/2))} + \left(\frac{\mathrm{Vol}_h(B_h(p_0,k_1))}{\mathrm{Vol}_h(B_h(p_0,k_1/2))}\right)^2\right\}(1+\sup_{z\in B_h(y,k_1)}|\rho(p_0,z)-\overline{\rho}(y)|)\\
\le&(2k_1+1)C_2'\left\{\frac{V_{-k_0}(k_1)}{V_{-k_0}(k_1/2)} + \left(\frac{V_{-k_0}(k_1)}{V_{-k_0}(k_1/2)}\right)^2\right\}\\
\le& C_1' \quad \forall y\in M
\end{aligned}
\tag{1.23}
$$

for some cosntants $C_1'>0$ and $C_2'>0$. By (1.16), (1.21) and (1.22), $\overline{\rho}\in C^2(M)$ and the lemma follows.

For any $a\ge 1$, let

$$\overline{\rho}_a(y)=\overline{\rho}(y)(1-\eta(\overline{\rho}(y)/a)). \tag{1.24}$$

Then

$$\overline{\rho}_a(y)=0 \quad \forall\overline{\rho}(y)\le a/2. \tag{1.25}$$

Since $\overline{\rho}_a(y) = \overline{\rho}(y)$ for any $\overline{\rho}(y) \ge a$, by (1.13),

$$\rho(p_0, y) - k_1 \le \overline{\rho}_a(y) \le \rho(p_0, y) + k_1 \quad \forall \overline{\rho}(y) \ge a. \tag{1.26}$$

Since

$$\widetilde{\nabla}_\beta \overline{\rho}_a = (1 - \eta(\overline{\rho}/a))\widetilde{\nabla}_\beta \overline{\rho} - \overline{\rho}\eta' \cdot (\widetilde{\nabla}_\beta \overline{\rho}/a)$$

and

$$\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \overline{\rho}_a = (1 - \eta(\overline{\rho}/a))\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \overline{\rho} - 2(\eta'/a)\widetilde{\nabla}_\alpha \overline{\rho} \widetilde{\nabla}_\beta \overline{\rho} - \overline{\rho}(\eta'/a)\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \overline{\rho} - \overline{\rho}(\eta''/a^2)\widetilde{\nabla}_\alpha \overline{\rho} \widetilde{\nabla}_\beta \overline{\rho},$$

by Lemma 1.4 we have the following result.

Corollary 1.5. *$\overline{\rho}_a \in C^2(M)$ for any $a \ge 1$ and there exists a constant $C_2 > 0$ such that*

$$\begin{cases} |\widetilde{\nabla} \overline{\rho}_a(y)|_h \le C_2 \quad \forall y \in M, a \ge 1 \\ |\widetilde{\nabla}^2 \overline{\rho}_a(y)|_h \le C_2 \quad \forall y \in M, a \ge 1. \end{cases}$$

For any $a \ge 1$, let

$$\psi_a(y) = 4\sqrt{k_0}\,\overline{\rho}_a(y)$$

and

$$\begin{cases} \overset{a}{h}_{\alpha\beta} = e^{\psi_a} h_{\alpha\beta} \\ \overset{a}{g}_{ij} = e^{\psi_a} g_{ij}. \end{cases}$$

Note that by the results of [CGT] and [CLY], there exists a constant $\delta_0 > 0$ depending on k_0 and the injectivity radius of (N, h) at p_0 such that

$$inj_h(y) \ge \delta_0\, e^{-\sqrt{k_0}\rho(p_0, y)} \quad \forall y \in N \tag{1.27}$$

where $inj_h(y)$ is the injectivity radius of y in (N, h). Now by (1.13) and (1.26),

$$\begin{aligned} & \rho(p_0, y) - k_1 \le \overline{\rho}_a(y) \le \rho(p_0, y) + k_1 \quad \forall \rho(p_0, y) \ge a + k_1, a \ge 1 \\ \Rightarrow \quad & \overline{\rho}_a(y) \ge 2\rho(p_0, y)/3 \quad \forall \rho(p_0, y) \ge 4a/3, a \ge \max(1, 3k_1) \end{aligned} \tag{1.28}$$

where $k_1 = \pi/4\sqrt{k_0}$. By (1.27), (1.28), and an argument similar to the proof on P.125 of [CZ], for any $a \ge \max(1, 3\pi/4\sqrt{k_0})$,

$$i_a = \mathrm{inj}\,(N, h^a) > 0. \tag{1.29}$$

We will now let $\overset{a}{\mathrm{Rm}}$, $\overset{a}{\widetilde{\mathrm{Rm}}}$, $\overset{a}{R}_{ijkl}$, $\overset{a}{\widetilde{R}}_{\alpha\beta\gamma\delta}$, etc. be the Riemann curvature, Riemannian curvature tensor, etc. of $\overset{a}{g}_{ij}$ and $\overset{a}{h}_{\alpha\beta}$ respectively.

Lemma 1.6. *There exists a constant $C_3 > 0$ such that*

$$|\overset{a}{\widetilde{Rm}}|_{h^a} \le C_3 e^{-\psi_a} \quad \text{in } N \quad \forall a \ge \max(1, 3\pi/4\sqrt{k_0}).$$

Proof. Let $a \ge \max(1, 3\pi/4\sqrt{k_0})$. By direct computation (cf. (2.9) of [CZ] and (13) of P.299 of [S1]),

$$\begin{aligned}\overset{a}{\widetilde{R}}_{\alpha\beta\gamma\delta} =& e^{\psi_a}\widetilde{R}_{\alpha\beta\gamma\delta} + \frac{e^{\psi_a}}{4}\{|\widetilde{\nabla}\psi_a|^2(h_{\alpha\delta}h_{\beta\gamma} - h_{\alpha\gamma}h_{\beta\delta}) + (2\widetilde{\nabla}_\alpha\widetilde{\nabla}_\delta\psi_a - \widetilde{\nabla}_\alpha\psi_a\widetilde{\nabla}_\delta\psi_a)h_{\beta\gamma}\\ &+ (2\widetilde{\nabla}_\beta\widetilde{\nabla}_\gamma\psi_a - \widetilde{\nabla}_\beta\psi_a\widetilde{\nabla}_\gamma\psi_a)h_{\alpha\delta} - (2\widetilde{\nabla}_\beta\widetilde{\nabla}_\delta\psi_a - \widetilde{\nabla}_\beta\psi_a\widetilde{\nabla}_\delta\psi_a)h_{\alpha\gamma}\\ &- (2\widetilde{\nabla}_\alpha\widetilde{\nabla}_\gamma\psi_a - \widetilde{\nabla}_\alpha\psi_a\widetilde{\nabla}_\gamma\psi_a)h_{\beta\delta}\}. \end{aligned} \tag{1.30}$$

Hence by Corollary 1.5 and (1.30),

$$\begin{aligned}|\overset{a}{\widetilde{\mathrm{Rm}}}|_{h^a} \le& Ce^{-\psi_a}(|\widetilde{Rm}|_h + |\widetilde{\nabla}^2\psi_a|_h + |\widetilde{\nabla}\psi_a|_h^2)\\ =& Ce^{-\psi_a}(|\widetilde{Rm}|_h + 4|\widetilde{\nabla}^2\overline{\rho}_a|_h + 16|\widetilde{\nabla}\overline{\rho}_a|_h^2)\\ \le& C_3' e^{-\psi_a}(|\widetilde{Rm}|_h + 1)\end{aligned}$$

for some constants $C > 0, C_3' > 0$, and the lemma follows.

By a similar argument as the proof of Lemma 1.6 we have the following result.

Corollary 1.7. *There exists a constant $C_4 > 0$ such that*

$$|\overset{a}{Rm}|_{g^a} \le C_4 e^{-\psi_a} \quad \text{in } M \times (0,T) \quad \forall a \ge \max(1, 3\pi/4\sqrt{k_0}).$$

Section 2

In this section we will construct solutions $\overset{a}{F} : (M, \overset{a}{g}) \to (N, \overset{a}{h})$ of

$$\frac{\partial \overset{a}{F}}{\partial t} = \overset{a}{\Delta}_{\overset{a}{g}(t), \overset{a}{h}} \overset{a}{F} \tag{2.1}$$

in bounded cylindrical domains with Dirichlet boundary condition where

$$\overset{a}{\Delta}_{\overset{a}{g}(t), \overset{a}{h}} F = \Delta_{\overset{a}{g}(t)} F + \overset{a}{g}{}^{ij}(x,t)\overset{a}{\widetilde{\Gamma}}{}^\alpha_{\beta,\gamma}(\overset{a}{F}(x,t))\frac{\partial \overset{a}{F}{}^\beta}{\partial x^i}\frac{\partial \overset{a}{F}{}^\gamma}{\partial x^j}$$

and

$$\Delta_{\overset{a}{g}(t)} F^\alpha = \overset{a}{g}_{ij} \overset{a}{\nabla}_i \overset{a}{\nabla}_j F^\alpha.$$

We will construct solution for the approximate problem

$$\begin{cases} \dfrac{\partial \overset{a}{F}}{\partial t} = \overset{a}{\Delta}_{\overset{a}{g}(t),\overset{a}{h}} F \quad \text{ in } M \times (0, T_1) \\ F(x,0) = f(x) \quad \text{ in } M \end{cases} \tag{2.2}$$

of (0.2) in $M \times (0, T_1)$. We will prove that the solutions of (2.2) has a subsequence that converges to a solution of (0.2) as $a \to \infty$. We will now assume that $a \geq \max(1, 3\pi/4\sqrt{k_0})$ for the rest of the paper. Note that as before the Levi-Civita connections $\overset{a}{\nabla}$ and $\overset{a}{\widetilde{\nabla}}$ on $(M, \overset{a}{g})$ and $(N, \overset{a}{h})$ respectively induce a natural connection $\overset{a}{\nabla}$ on $T^*M^{\otimes p} \otimes F^{-1}TN$ for any $p \geq 1$. We will write $\overset{a}{\Delta}_t$ for $\Delta_{\overset{a}{g}(t)} = \overset{a}{g}_{ij} \overset{a}{\nabla}_i \overset{a}{\nabla}_j$.

Note that

$$\sup_M |\overset{a}{\nabla} f|_{\overset{a}{g}(0),\overset{a}{h}} = \sup_M |\nabla f|_{g(0),h} = K_1 \quad \forall a \geq 1. \tag{2.3}$$

By direct computation,

$$\overset{a}{\Gamma}{}_{ij}^k = \Gamma_{ij}^k + \frac{1}{2}\{\delta_i^k \nabla_j \psi_a + \delta_j^k \nabla_i \psi_a - g^{kl} g_{ij} \nabla_l \psi_a\}. \tag{2.4}$$

Hence

$$\begin{aligned}
\overset{a}{\nabla}_i \overset{a}{\nabla}_j f^\alpha =& \nabla_i \nabla_j f^\alpha + (\Gamma_{ij}^k - \overset{a}{\Gamma}{}_{ij}^k) \nabla_k f^\alpha + (\overset{a}{\widetilde{\Gamma}}{}_{\beta\gamma}^\alpha - \Gamma_{\beta\gamma}^\alpha) \nabla_i f^\beta \nabla_j f^\gamma \\
=& \nabla_i \nabla_j f^\alpha - \frac{1}{2}\{\delta_i^k \nabla_j \psi_a + \delta_j^k \nabla_i \psi_a - g^{kl} g_{ij} \nabla_l \psi_a\} \nabla_k f^\alpha \\
&+ \frac{1}{2}\{\delta_\beta^\alpha \nabla_\gamma \psi_a + \delta_\gamma^\alpha \nabla_\beta \psi_a - h^{\alpha\delta} h_{\beta\gamma} \nabla_\delta \psi_a\} \nabla_i f^\beta \nabla_j f^\gamma \\
=& \nabla_i \nabla_j f^\alpha - \frac{1}{2}\{\nabla_i f^\alpha \nabla_j \psi_a + \nabla_j f^\alpha \nabla_i \psi_a - g^{kl} g_{ij} \nabla_k f^\alpha \nabla_l \psi_a\} \\
&+ \frac{1}{2}\{\nabla_i f^\alpha \nabla_j f^\gamma \nabla_\gamma \psi_a + \nabla_j f^\alpha \nabla_i f^\beta \nabla_\beta \psi_a - h^{\alpha\delta} h_{\beta\gamma} \nabla_i f^\beta \nabla_j f^\gamma \nabla_\delta \psi_a\}
\end{aligned} \tag{2.5}$$

By (0.10), (0.11), (2.5) and Corollary 1.5, there exists a constant $C > 0$ such that

$$|\overset{a}{\nabla}{}^2 f(x)|_{\overset{a}{g}(0),\overset{a}{h}} \leq C e^{-\frac{1}{2}\psi_a(x)} \leq C \quad \forall x \in M, a \geq 1. \tag{2.6}$$

By Lemma 1.6, Lemma 1.1 for $(N, \overset{a}{h})$, and an argument similar to the proof of Lemma 2.8 of [CZ] we have

Lemma 2.1. *For any $a \geq \max(1, 3\pi/4\sqrt{k_0})$, there exist constants $0 < T_2 \leq T$ and $C_5 > 0$ depending on n, a, and k_0 such that for any $k > 0$, $0 < T_1 \leq T$, and solution*

$\overset{a}{F}_k : (B_k, g(t)) \to (N, h_{\alpha\beta})$, $\overset{a}{F}_k \in C^{2+\frac{1}{2},1+\frac{1}{2}}(\overline{B}_k \times [0, T_1))$, *of*

$$\begin{cases} \dfrac{\partial \overset{a}{F}_k}{\partial t} = \overset{a}{\Delta}_{\overset{a}{g}(t),\overset{a}{h}} \overset{a}{F}_k \quad \text{in } Q_k^{T_1} \\ \overset{a}{F}_k(x,t) = f(x) \quad \text{on } \partial B_k \times (0, T_1) \\ \overset{a}{F}_k(x,0) = f(x) \quad \text{in } B_k, \end{cases} \tag{2.7}$$

we have

$$\rho(f(x), \overset{a}{F}_k(x,t)) \le C_5 \sqrt{t} \tag{2.8}$$

in $Q_k^{T_2'}$ *where* $T_2' = \min(T_2, T_1)$.

By Lemma 1.6, Lemma 1.7, (2.3), (2.6), an argument similar to the proof of Theorem 7.1 of Chapter VII of [LSU] (cf. P.245–246 of [S1]) and an argument similar to the proof of Lemma 2.9 of [CZ] but with Lemma 2.1 replacing Lemma 2.8 in the proof there we have

Lemma 2.2. *Let* $a \ge \max(1, 3\pi/4\sqrt{k_0})$ *and let* T_2 *be as given in Lemma 2.1. Then there exist constants* $0 < T_3 \le T_2$ *depending on* n, a, k_0, *and* f *such that for any* $k > 0$ *there exists a solution* $\overset{a}{F}_k \in C^{2+\frac{1}{2},1+\frac{1}{2}}(\overline{B}_k \times [0, T_3])$ *of (2.7) in* $Q_k^{T_3}$.

Remark. *Note that by Corollary 1.5 and an argument similar to the proof of Theorem 7.1 of Chapter VII of* [LSU] *for any solution* $\overset{a}{F}_k$ *of (2.7) in* $Q_k^{T_3}$ *given by Lemma 2.2, we have* $\overset{a}{F}_k \in C^{3,1+\frac{1}{2}}(\overline{B}_k \times (0, T_3])$.

Lemma 2.3. *(Section 6 of* [H3]*) Let* $\overset{a}{F}$ *be a solution of (2.1) in* $Q_k^{T'}$ *for some* $k > 0$ *and* $0 < T' \le T$. *Then* $\overset{a}{F}$ *satisfies*

$$\frac{\partial}{\partial t} |\overset{a}{\nabla} \overset{a}{F}|^2 = \overset{a}{\Delta}_t |\overset{a}{\nabla} \overset{a}{F}|^2 - 2|\overset{a}{\nabla}^2 \overset{a}{F}|^2 + 2\widetilde{\overset{a}{Rm}}(\overset{a}{\nabla} \overset{a}{F}, \overset{a}{\nabla} \overset{a}{F}, \overset{a}{\nabla} \overset{a}{F}, \overset{a}{\nabla} \overset{a}{F}) \quad \text{in } Q_k^{T'}$$

where

$$\widetilde{\overset{a}{Rm}}(\overset{a}{\nabla} \overset{a}{F}, \overset{a}{\nabla} \overset{a}{F}, \overset{a}{\nabla} \overset{a}{F}, \overset{a}{\nabla} \overset{a}{F}) = \overset{a}{g}^{ik} \overset{a}{g}^{jl} \overset{a}{\widetilde{R}}_{pqrs} \overset{a}{\nabla}_i \overset{a}{F}^p \overset{a}{\nabla}_j \overset{a}{F}^q \overset{a}{\nabla}_k \overset{a}{F}^r \overset{a}{\nabla}_l \overset{a}{F}^s.$$

We next observe that by Lemma 1.6 and Lemma 2.3,

$$\frac{\partial}{\partial t} |\overset{a}{\nabla} \overset{a}{F}|^2 \le \overset{a}{\Delta}_t |\overset{a}{\nabla} \overset{a}{F}|^2 - 2|\overset{a}{\nabla}^2 \overset{a}{F}|^2 + C_3 |\overset{a}{\nabla} F|^4 \tag{2.9}$$

in $Q_k^{T'}$ for any solution $\overset{a}{F}$ of (2.1) in $Q_k^{T'}$ and any $a \ge \max(1, 3\pi/4\sqrt{k_0})$ where $C_3 > 0$ is the constant given by Lemma 1.6. Then by (2.3), (2.9), Lemma 2.1 and an argument similar to the proof of Lemma 2.11 of [CZ] but with (2.9) replacing (2.30) in the proof there we have

Lemma 2.4. *Let T_3 be as in Lemma 2.2. Let $k_0 > 0$ be given by (0.8). Then there exist constants $0 < \delta_1 < 1$, $C_6 > 0$, and $0 < T_4 \le T_3$ depending on n, a, k_0, and K_1 such that for any solution $\overset{a}{F}_k$ of (2.7) given by Lemma 2.2 and any $B_{\overset{a}{g}(0)}(x_0, \delta_1) \subset B_k$ we have*

$$|\overset{a}{\nabla}\overset{a}{F}_k|(x,t) \le C_6 \tag{2.10}$$

holds in $B_{\overset{a}{g}(0)}(x_0, 3\delta_1/4) \times [0, T_4]$ for any $k > 0$ and $a \ge \max(1, 3\pi/4\sqrt{k_0})$.

Theorem 2.5. *Let $a \ge \max(1, 3\pi/4\sqrt{k_0})$ and let T_4, C_6, be given by Lemma 2.4. Then there exists a solution $\overset{a}{F} \in C^{2+\frac{1}{2},1+\frac{1}{2}}(M \times [0,T_4]) \cap C^{3,1+\frac{1}{2}}(M \times (0,T_4])$ of (2.2) in $M \times (0,T_4)$ which satisfies*

$$|\overset{a}{\nabla}\overset{a}{F}|(x,t) \le C_6 \quad \text{on } M \times [0,T_4]. \tag{2.11}$$

Proof. By Lemma 2.2 for any $k \in \mathbb{Z}^+$ there exists a solution $\overset{a}{F}_k \in C^{2+\frac{1}{2},1+\frac{1}{2}}(\overline{B}_k \times [0,T_4])$ of (2.7) in $Q_k^{T_4}$. By Lemma 2.4 (2.10) holds. Hence for any $k' \in \mathbb{Z}^+$ the sequence $\{|\overset{a}{\nabla}\overset{a}{F}_k|\}_{k=k'}^{\infty}$ is uniformly bounded on any compact subset of $\overline{B_{\overset{a}{g}(0)}(p_0, k'-1)} \times [0,T_4]$. Then by Corollary 1.5 and an argument similar to the proof of Theorem 7.1 of Chapter VII of [LSU] (cf. P.245–246 of [S1]) $\{\overset{a}{F}_k\}_{k=k'}^{\infty}$ is uniformly bounded in $C^{2+\frac{1}{2},1+\frac{1}{2}}(\overline{B_{\overset{a}{g}(0)}(p_0, k'-1)} \times [0,T_4])$.

By the Ascoli Theorem and a diagonalization argument the sequence $\{\overset{a}{F}_k\}_{k=1}^{\infty}$ has a convergent subsequence which we may assume without loss of generality to be the sequence itself such that $\overset{a}{F}_k$ converges uniformly in $C^{2+\frac{1}{2},1+\frac{1}{2}}(K)$ for any compact subset K of $M \times [0,T_4]$ to some function $\overset{a}{F}$ as $k \to \infty$. Then $\overset{a}{F} \in C^{2+\frac{1}{2},1+\frac{1}{2}}(M \times [0,T_4])$ is a solution of (2.2) in $M \times (0,T_4)$ which satisfies (2.11). By Corollary 1.5 and an argument similar to the proof of Theorem 7.1 of Chapter VII of [LSU] $\overset{a}{F} \in C^{3,1+\frac{1}{2}}(\overline{B}_k \times (0,T_3])$ and the theorem follows.

Theorem 2.6. *Let*

$$T_1 = \min(\log 2/(2C_3K_1^2), T) \tag{2.12}$$

where C_3 is given by Lemma 1.6. Then for any $a \ge \max(1, 3\pi/4\sqrt{k_0})$ there exists a solution $\overset{a}{F} \in C^{2+\frac{1}{2},1+\frac{1}{2}}(M \times [0,T_1]) \cap C^{3,1+\frac{1}{2}}(M \times (0,T_1])$ of (2.2) in $M \times (0,T_1)$ which satisfies

$$|\overset{a}{\nabla}\overset{a}{F}| \le 2K_1 \quad \text{on } M \times (0,T_1). \tag{2.13}$$

Proof. Let $a \ge \max(1, 3\pi/4\sqrt{k_0})$. Let T_4 and C_6 be as in Lemma 2.4. By Theorem 2.5 there exists a solution $\overset{a}{F} \in C^{2+\frac{1}{2},1+\frac{1}{2}}(M \times [0,T_4]) \cap C^{3,1+\frac{1}{2}}(M \times (0,T_4])$ of (2.2) in $M \times (0,T_4)$ which satisfies (2.11). For any $R_1 \ge 1$, let

$$u(x,t) = |\overset{a}{\nabla}\overset{a}{F}(x,t)|^2\widetilde{\phi}(x)$$

where $\widetilde{\phi}(x)=\phi(\rho(x)/R_1)$ with $\rho(x)=\rho(p_0,x)$. By the results in Chapter 1 of [SY] $\rho(p_0,\cdot)\in C^\infty(M\setminus(\mathrm{Cut}(p_0)\cup\{p_0\}))$ (cf. [GW1],[W1], [W2]). Hence $u(\cdot,t)\in C^\infty(M\setminus\mathrm{Cut}(p_0))$ for any $0<t\le T_4$. We first suppose that x is not a cut point of p_0. Then by (2.9),

$$
\begin{aligned}
u_t-\overset{a}{\Delta}_t u=&\widetilde{\phi}\left(\frac{\partial}{\partial t}-\overset{a}{\Delta}_t\right)|\overset{a}{\nabla}\overset{a}{F}|^2-2\overset{a}{\nabla}(|\overset{a}{\nabla}\overset{a}{F}|^2)\cdot\overset{a}{\nabla}\widetilde{\phi}-|\overset{a}{\nabla}\overset{a}{F}|^2\overset{a}{\Delta}_t\widetilde{\phi}\\
\le&(-2|\overset{a}{\nabla}{}^2\overset{a}{F}|^2+C_3|\overset{a}{\nabla}\overset{a}{F}|^4)\widetilde{\phi}+(2/R_1)|\phi'||\overset{a}{\nabla}(|\overset{a}{\nabla}\overset{a}{F}|^2)||\overset{a}{\nabla}\rho|_{\overset{a}{g}}\\
&+|\overset{a}{\nabla}\overset{a}{F}|^2\left(|\phi''|\frac{|\overset{a}{\nabla}\rho|_{\overset{a}{g}}^2}{R_1^2}+|\phi'|\frac{\overset{a}{\Delta}_t\rho}{R_1}\right).
\end{aligned}\tag{2.14}
$$

Let $c_2>0$ be as given by Lemma 1.2. Then by (1.7),

$$
|\overset{a}{\nabla}\rho|_{\overset{a}{g}}^2=e^{-\psi_a}g^{ij}(x,t)\frac{\partial\rho}{\partial x^i}\frac{\partial\rho}{\partial x^j}\le c_2g^{ij}(x,0)\frac{\partial\rho}{\partial x^i}\frac{\partial\rho}{\partial x^j}\le c_2.\tag{2.15}
$$

By (0.13) and (2.15),

$$
\frac{2}{R_1}|\phi'||\overset{a}{\nabla}(|\overset{a}{\nabla}\overset{a}{F}|^2)||\overset{a}{\nabla}\rho|_{\overset{a}{g}}\le\frac{4\sqrt{c_2}}{R_1}|\phi'||\overset{a}{\nabla}F||\overset{a}{\nabla}{}^2F|\le|\overset{a}{\nabla}{}^2F|^2\widetilde{\phi}+\frac{C_8}{R_1^2}|\overset{a}{\nabla}F|^2\tag{2.16}
$$

where

$$
C_8=4c_2\sup_{\mathbb{R}}(\phi'^2/\phi).
$$

By (2.4),

$$
\begin{aligned}
\overset{a}{\Delta}_t\rho=&\overset{a}{g}{}^{ij}\left\{\frac{\partial^2\rho}{\partial x^i\partial x^j}-\overset{a}{\Gamma}{}_{ij}^k\nabla_k\rho\right\}\\
=&e^{-\psi_a}\Delta_t\rho-\frac{1}{2}e^{-\psi_a}g^{ij}\left\{\delta_i^k\nabla_j\psi_a+\delta_j^k\nabla_i\psi_a-g^{kl}g_{ij}\nabla_l\psi_a\right\}\nabla_k\rho\\
=&e^{-\psi_a}\left(\Delta_t\rho+\frac{n-2}{2}g^{ij}\nabla_i\rho\nabla_j\psi_a\right)
\end{aligned}\tag{2.17}
$$

By an argument similar to the proof of Lemma 1.3,

$$
\Delta_t\rho\le nc_2\frac{1+\sqrt{k_0}\rho}{\rho}+C_9\sqrt{t}\tag{2.18}
$$

for some constant $C_9>0$ independent of a. By Lemma 1.2 and Corollary 1.5,

$$
|g^{ij}\nabla_i\rho\nabla_j\psi_a|\le|\nabla\rho|_{g(t)}|\nabla\psi_a|_{g(t)}\le c_2|\nabla\rho|_{g(0)}|\widetilde{\nabla}\psi_a|_h\le C_{10}\tag{2.19}
$$

for some constant $C_{10} > 0$. By (2.14), (2.15), (2.16), (2.17), (2.18) and (2.19),

$$u_t - \overset{a}{\Delta}_t u \le (-|\overset{a}{\nabla}{}^2 \overset{a}{F}|^2 + C_3 |\overset{a}{\nabla} \overset{a}{F}|^4)\widetilde{\phi} + \frac{C_{11}}{R_1}|\overset{a}{\nabla} \overset{a}{F}|^2 \le C_3 |\overset{a}{\nabla} \overset{a}{F}|^2 u + \frac{C_{11}}{R_1}|\overset{a}{\nabla} \overset{a}{F}|^2 \tag{2.20}$$

in $M \times (0, T_4)$ for some constant $C_{11} > 0$. By (2.11) and (2.20),

$$\begin{aligned} & u_t - \overset{a}{\Delta}_t u \le C_3 C_6^2 u + \frac{C_{11}}{R_1^2} C_6^2 && \text{in } (M \setminus \mathrm{Cut}(p_0)) \times (0, T_4] \\ \Rightarrow \quad & \left(\frac{\partial}{\partial t} - \overset{a}{\Delta}_t\right)(e^{-C_3 C_6^2 t} u) \le \frac{C_{11} C_6^2}{R_1} e^{-C_3 C_6^2 t} < \frac{2 C_{11} C_6^2}{R_1} && \text{in } (M \setminus \mathrm{Cut}(p_0)) \times (0, T_4]. \end{aligned} \tag{2.21}$$

Let

$$q(y, t) = e^{-C_3 C_6^2 t} u(x, t) - (2 C_{11} C_6^2 / R_1) t.$$

Then by (2.21),

$$q_t < \overset{a}{\Delta}_t q \tag{2.22}$$

in $(M \setminus \mathrm{Cut}(p_0)) \times (0, T_4]$. Suppose there exists $(x_0, t_0) \in B(R_1) \times (0, T_4]$ such that

$$q(x_0, t_0) = \max_{\overline{B(R_1)} \times [0, T_4]} q(x, t).$$

Suppose first that $x_0 \in M \setminus \mathrm{Cut}(p_0)$. Then

$$\begin{aligned} & \frac{\partial^2 q}{\partial x^i \partial x^j}(x_0, t_0) \le 0, \frac{\partial q}{\partial t}(x_0, t_0) \ge 0, \frac{\partial q}{\partial x^i}(x_0, t_0) = 0 \quad \forall i, j = 1, 2, \cdots, n \\ \Rightarrow \quad & q_t(x_0, t_0) \ge \overset{a}{\Delta}_t q(x_0, t_0) \end{aligned} \tag{2.23}$$

which contradict (2.22). Hence $x_0 \in \mathrm{Cut}(p_0)$. Let γ be a minimal geodesic in $(M, g(0))$ joining p_0 and x_0. Let

$$0 < \varepsilon < \min(\rho(p_0, x_0), R_1/5).$$

We choose a point x_ε along the geodesic γ such that $\rho(p_0, x_\varepsilon) = \varepsilon$. Then x_0 is not a cut point of x_ε. Hence there exist a constant $\delta > 0$ such that $B(x_0, \delta) \subset B(p_0, R_1) \setminus \mathrm{Cut}(x_\varepsilon)$. Let

$$u_\varepsilon(x, t) = |\overset{a}{\nabla} \overset{a}{F}(x, t)|^2 \phi((\varepsilon + \rho(x_\varepsilon, x))/R_1) \quad \forall x \in M, 0 \le t \le T_4.$$

and

$$q_\varepsilon(x, t) = e^{-C_3 C_6^2 t} u_\varepsilon(x, t) - (2 C_{11} C_6^2 / R_1) t.$$

By (2.21) and a similar argument as before q_ε satisfies (2.22) in $B(x_0, \delta) \times (0, T_4]$. Since

$$\rho(p_0, x) \le \varepsilon + \rho(x_\varepsilon, x) \quad \text{ in } B(x_0, \delta)$$

and

$$\rho(p_0, x_0) = \rho(p_0, x_\varepsilon) + \rho(x_\varepsilon, x_0) = \varepsilon + \rho(x_\varepsilon, x_0),$$

we have

$$q_\varepsilon(x,t) \le q(x,t) \quad \text{in } B(x_0,\delta)\times(0,T_4]$$

and

$$q_\varepsilon(x_0,t_0) = q(x_0,t_0).$$

Hence q_ε attains its maximum in $B_h(x_0,\delta)\times(0,T_4]$ at (x_0,t_0). Thus q_ε also satisfies (2.23). This contradicts (2.22) for q_ε. Hence no such interior maximum point (x_0,t_0) exists and

$$\begin{aligned}
& q(x,t) \le \max_{\partial_p Q_{R_1}^{T_4}} q \le \sup_M |\nabla f|^2 = K_1^2 && \text{in } Q_{R_1}^{T_4} \quad \forall R_1 \ge 1\\
\Rightarrow\quad & e^{-C_3C_6^2 t}|\overset{a}{\nabla}\overset{a}{F}(x,t)|^2\phi(\rho(p_0,x)/R_1) \le K_1^2 + (2C_{11}C_6^2/R_1)t && \text{in } Q_{R_1}^{T_4} \quad \forall R_1 \ge 1\\
\Rightarrow\quad & |\overset{a}{\nabla}\overset{a}{F}(x,t)| \le K_1 e^{\frac{C_3C_6^2}{2}t} && \forall x\in M, 0\le t\le T_4 \quad \text{as } R_1\to\infty.
\end{aligned} \tag{2.24}$$

Let $T' = \min(2\log 2/C_3C_6^2, T_4)$. By (2.24), (2.13) holds in $M\times[0,T']$. Let

$$\begin{aligned}
T_1' = \sup\{T''\in(0,T) : \exists \text{ a solution of (2.2) in } M\times[0,T'') \text{ such that (2.13) holds}\\
\text{in } M\times(0,T'')\}.
\end{aligned}$$

Then $T_1' \ge T'$. We claim that $T_1' \ge T_1$. Suppose not. Then $T_1' < T_1$. Since (2.13) holds in $M\times[0,T_1']$, by an argument similar to the proof of (2.24) but with (2.13) replacing (2.11) in the proof,

$$|\overset{a}{\nabla}\overset{a}{F}(x,t)| \le K_1 e^{\frac{C_3(2K_1)^2}{2}t} = K_1 e^{2C_3K_1^2 t} \le K_1 e^{2C_3K_1^2T_1'} < 2K_1 \quad \forall x\in M, 0\le t\le T_1'. \tag{2.25}$$

By Theorem 2.5 there exists a solution $\overset{a}{F}_1 \in C^{2+\frac12,1+\frac12}(M\times[0,T_0])\cap C^{3,1+\frac12}(M\times(0,T_0])$ of (2.2) in $M\times(0,T_0)$ with initial value $\overset{a}{F}(x,T_1')$ for some constant $T_0\in(0,T-T_1')$ which satisfies

$$|\overset{a}{\nabla}\overset{a}{F}_1| \le \widetilde{C}_6 \quad \text{on } M\times[0,T_0]$$

for some constant $\widetilde{C}_6>0$. We extend $\overset{a}{F}$ to a solution of (2.1) in $M\times(0,T_1+T_0)$ by setting $\overset{a}{F}(x,t) = F_1(x,t-T_1')$ for any $x\in M$, $T_1'\le t\le T_1'+T_0$. By (2.25) and an argument similar to the proof of (2.24),

$$|\overset{a}{\nabla}\overset{a}{F}_1(x,t)| \le e^{\frac{C_3\widetilde{C}_6^2}{2}t}|\overset{a}{\nabla}\overset{a}{F}(x,T_1')| \le K_1 e^{2C_3K_1^2T_1'}e^{\frac{C_3\widetilde{C}_6^2}{2}t} \quad \forall x\in M, 0\le t\le T_0 \tag{2.26}$$

Let $\delta_2 = -2C_3K_1^2T_1' + \log 2$. Then $\delta_2>0$. Let $T_0' = \min(2\delta_2/(C_3\widetilde{C}_6^2), T_0)$. Then by (2.26),

$$\begin{aligned}
& |\overset{a}{\nabla}\overset{a}{F}_1(x,t)| \le 2K_1 \quad \text{on } M\times[0,T_0]\\
\Rightarrow\quad & |\overset{a}{\nabla}\overset{a}{F}(x,t)| \le 2K_1 \quad \text{on } M\times[0,T_1'+T_0].
\end{aligned}$$

This contradicts the maximality of T_1'. Hence $T_1' \ge T_1$ and the theorem follows.

Theorem 2.7. *Let $f : (M, g(0)) \to (N, h)$ be a given diffeomorphism satisfying (0.10) and (0.11) and let T_1 be given by (2.12). Then there exists a smooth solution $F(\cdot, t) : (M, g(t)) \to (N, h)$ to the following Ricci harmonic flow:*

$$\begin{cases} \dfrac{\partial F}{\partial t} = \Delta_{g(t),h} F & \text{in } M \times (0, T_1) \\ F(x, 0) = f(x) & \text{in } M \end{cases} \tag{2.27}$$

which satisfies

$$|\nabla F| \le 2K_1 \quad \text{in } M \times [0, T_1] \tag{2.28}$$

and

$$\sup_{M\times[0,T_1)} |\nabla^m F| \le C_m t^{-\frac{m-2}{2}} \qquad \text{in } M \times [0, T_1] \quad \forall m \ge 2 \tag{2.29}$$

for some constants $C_m > 0$ depending on k_0, K_1 and K_2.

Proof. Let $a_1 = \max(1, 3\pi/4\sqrt{k_0})$. For any $a \ge a_1$, let $\overset{a}{F} \in C^{2+\frac{1}{2},1+\frac{1}{2}}(M \times [0, T_1]) \cap C^{3,1+\frac{1}{2}}(M \times (0, T_1])$ be the solution of (2.2) in $M \times (0, T_1)$ given by Theorem 2.6 which satisfies (2.13). By an argument similar to the proof of Theorem 7.1 of Chapter VII of [LSU] (cf. P.245–246 of [S1]) $\{\overset{a}{F}\}_{a\ge a_1}$ is uniformly bounded in $C^{2+\frac{1}{2},1+\frac{1}{2}}(K)$ for any compact subset K of $M \times [0, T_1]$. By the Ascoli Theorem and a diagonalization argument the sequence $\{\overset{a}{F}\}_{a\ge a_1}$ has a convergent subsequence $\{\overset{a_i}{F}\}_{i=1}^{\infty}$ that converges uniformly in $C^{2+\frac{1}{2},1+\frac{1}{2}}(K)$ for any compact subset K of $M \times [0, T_1]$ to some function F as $a_i \to \infty$. Then $F \in C^{2+\frac{1}{2},1+\frac{1}{2}}(M \times [0, T_1])$.

Note that since $a \ge \max(1, 3\pi/4\sqrt{k_0})$, by (1.13) and (1.25),

$$\overline{\rho}_a(y) = 0 \quad \forall \rho(p_0, y) \le \frac{a}{2} - k_1 \quad \Rightarrow \quad \psi_a(y) = 1 \quad \forall \rho(p_0, y) \le \frac{a}{2} - k_1.$$

Thus F is a solution of (2.27) in $M \times (0, T_1)$. By (2.13) F satisfies (2.28). By a bootrap argument and an argument similar to the proof of Theorem 7.1 of Chapter VII of [LSU] and that of P.245–246 of [S1] $F \in C^\infty(M \times (0, T_1))$. By (2.28) and an argument similar to the proof of Lemma 2.12 of [CZ] but with $\phi^a \equiv 0$ and (2.28) replacing Lemma 2.11 in the proof there we get (2.29) and the theorem follows.

Note that by (0.8) and Section 6 of [H3],

$$\frac{\partial}{\partial t}|\nabla F|^2 \le \Delta_t |\nabla F|^2 - 2|\nabla^2 F|^2 + k_0 |\nabla F|^4 \tag{2.30}$$

holds for any solution of (2.27). Hence by (2.30), Theorem 2.7 and an argument similar to the proof of Theorem 2.6 we have the following theorem.

Theorem 2.8. *Let $f : (M, g(0)) \to (N, h)$ be a given diffeomorphism satisfying (0.10) and (0.11) and let $T_1 = \min(\log 2/(2k_0 K_1^2), T)$. Then (2.27) has a smooth solution $F(\cdot, t) : (M, g(t)) \to (N, h)$ in $M \times (0, T_1)$ which satisfies (2.28) and (2.29).*

Let $I : (M, g(0)) \to (N, h)$ be the identity map. By direct computation,

$$|\nabla I|_{g(0),h} = (g^{ij}(x,0) g_{\alpha\beta}(x,0) \delta_i^\alpha \delta_j^\beta)^{\frac{1}{2}} = \sqrt{n}$$

and

$$|\nabla^2 I|_{g(0),h} = 0.$$

Hence by Theorem 2.8 we have the following result.

Theorem 2.9. *Let*

$$T_1 = \min(\log 2/(2k_0 n), T).$$

Then there exists a smooth solution $F(\cdot, t) : (M, g(t)) \to (N, h)$ to the following Ricci harmonic flow:

$$\begin{cases} \dfrac{\partial F}{\partial t} = \Delta_{g(t),h} F & \text{in } M \times (0, T_1) \\ F(x, 0) = x & \text{in } M \end{cases} \tag{2.31}$$

which satisfies (2.28) and (2.29) with $K_1 = \sqrt{n}$ for some constants $C_m > 0$ depending on k_0.

Section 3

In this section we will prove the uniqueness of solutions of Ricci flow on complete noncompact manifolds with bounded curvature.

Lemma 3.1. *(cf. Proposition 3.1 of* [CZ]*) Let F and T_1 be as given by Theorem 2.9. Let $h = g(0)$ and let $\overline{h}(t) = F^*(h) = (F(\cdot, t))^*(h)$ be the pull back metric of h on M by F. Then there exist a constant $C_1 > 0$ and a constant $0 < T_2 \le T_1$ depending only on k_0 such that*

$$\frac{1}{C_1} \overline{h}_{ij} \le g_{ij} \le C_1 \overline{h}_{ij} \quad \text{in } M \times [0, T_2] \tag{3.1}$$

and

$$|\overline{\nabla}^k g|_{\overline{h}} \le C_k t^{-\frac{k-1}{2}} \quad \text{in } M \times [0, T_2] \quad \forall k \in \mathbb{Z}^+ \tag{3.2}$$

for some constants C_k where $\overline{\nabla}$ is the Levi-Civita connection on M with respect to $\overline{h}$.

Proof. (3.2) and the first inequality on the the left hand side of (3.1) is proved in Proposition 3.1 of [CZ]. However the proof of the second inequality on the right hand side of (3.1) in [CZ] is questionable. For the sake of completeness we will give a correct proof here.

By direct computation,

$$\begin{aligned} \frac{\partial \overline{h}_{ij}}{\partial t} =& \nabla_t (h_{\alpha\beta} \nabla_i F^\alpha \nabla_j F^\beta) = h_{\alpha\beta} (\nabla_t \nabla_i F^\alpha)(\nabla_j F^\beta) + h_{\alpha\beta} (\nabla_i F^\alpha)(\nabla_t \nabla_j F^\beta) \\ =& h_{\alpha\beta} \nabla_i (F^\alpha)_t \nabla_j F^\beta + h_{\alpha\beta} \nabla_i F^\alpha \nabla_j (F^\beta)_t. \end{aligned} \tag{3.3}$$

Now

$$
\begin{aligned}
&h_{\alpha\beta}\nabla_i(F^\alpha)_t\nabla_j F^\beta\\
=&h_{\alpha\beta}\nabla_i(\Delta_{g(t),h}F^\alpha)\nabla_j F^\beta\\
=&h_{\alpha\beta}g^{i'j'}(\nabla_i\nabla_{i'}\nabla_{j'}F^\alpha)(\nabla_j F^\beta)\\
=&h_{\alpha\beta}g^{i'j'}(\nabla_{i'}\nabla_i\nabla_{j'}F^\alpha-R_{i'ij'l}g^{lk}\nabla_k F^\alpha+\widetilde{R}_{pqrs}h^{\alpha r}\nabla_i F^p\nabla_{i'}F^q\nabla_{j'}F^s)(\nabla_j F^\beta)\\
=&h_{\alpha\beta}\Delta_t(\nabla_i F^\alpha)\nabla_j F^\beta-h_{\alpha\beta}g^{lk}R_{ik}\nabla_l F^\alpha\nabla_j F^\beta+\widetilde{R}_{\alpha\beta\gamma\delta}g^{kl}\nabla_j F^\alpha\nabla_k F^\beta\nabla_i F^\gamma\nabla_l F^\delta.
\end{aligned}
\tag{3.4}
$$

By (3.3) and (3.4),

$$
\begin{aligned}
\frac{\partial\overline{h}_{ij}}{\partial t}=&h_{\alpha\beta}\nabla_i F^\alpha\Delta_t(\nabla_j F^\beta)+h_{\alpha\beta}\Delta_t(\nabla_i F^\alpha)\nabla_j F^\beta-h_{\alpha\beta}g^{lk}R_{ik}\nabla_l F^\alpha\nabla_j F^\beta\\
&-h_{\alpha\beta}g^{lk}R_{jk}\nabla_l F^\alpha\nabla_i F^\beta+2g^{lk}\widetilde{R}_{\alpha\beta\gamma\delta}\nabla_i F^\alpha\nabla_k F^\beta\nabla_j F^\gamma\nabla_l F^\delta.
\end{aligned}
\tag{3.5}
$$

Hence by (3.5), (2.29), (2.30) and Theorem 2.9,

$$
\begin{aligned}
&\left|\frac{\partial\overline{h}_{ij}}{\partial t}\right|_{g(t)}\le C\sup_M(|\nabla F||\nabla^3F|+|\nabla F|^2+|\nabla F|^4)\le\frac{C}{\sqrt{t}}\quad\text{in } M\times(0,T_1)\\
\Rightarrow\quad&-\frac{C_1}{\sqrt{t}}g_{ij}\le\frac{\partial\overline{h}_{ij}}{\partial t}\le\frac{C_1}{\sqrt{t}}g_{ij}\quad\text{in } M\times(0,T_1)
\end{aligned}
\tag{3.6}
$$

for some constant $C_1>0$. By Lemma 1.2 there exists constants $c_1,c_2>0$ such that (1.7) holds. By (1.7) and (3.6),

$$
\begin{aligned}
\frac{\partial\overline{h}_{ij}}{\partial t}(x,t)\ge&-\frac{c_2C_1}{\sqrt{t}}g_{ij}(x,0)\quad\text{in } M\times(0,T)\\
\Rightarrow\quad\overline{h}_{ij}(x,t)\ge&h_{ij}(x,0)-2c_2C_1\sqrt{t}g_{ij}(x,0)\quad\text{in } M\times(0,T)\\
=&g_{ij}(x,0)-2c_2C_1\sqrt{t}g_{ij}(x,0)\quad\text{in } M\times(0,T)\\
\ge&(1-2c_2C_1\sqrt{t})g_{ij}(x,0)\quad\text{in } M\times(0,T).
\end{aligned}
\tag{3.7}
$$

Let $T_2=\min(T_1,1/(16c_2^2C_1^2))$. Then by (3.7) and Lemma 1.2,

$$
\overline{h}_{ij}(x,t)\ge g_{ij}(x,0)/2\ge(1/2c_2)g_{ij}(x,t)\quad\forall x\in M,0\le t\le T_2
$$

and the lemma follows.

By Theorem 2.9, Lemma 3.1, and an argument similar to the proof of Proposition 3.2 of [CZ] we have the following result.

Lemma 3.2. *(cf. Proposition 3.1 and Proposition 3.2 of* [CZ]*) Let F and T_1 be as given by Theorem 2.9. Then there exists a constant $0 < T_2 = T_2(g) \le T_1$ depending only on k_0 such that $F(\cdot,t) : (M, g(t)) \to (N, h)$ is a diffeomorphism for any $0 \le t < T_2$. Moreover if $\hat{g}(t)$ is given by (0.4), then there exists a constant $C_1 > 0$ such that*

$$\frac{1}{C_1} h_{\alpha\beta} \le \hat{g}_{\alpha\beta} \le C_1 h_{\alpha\beta} \quad \text{in } N \times [0, T_2] \tag{3.8}$$

and

$$|\hat{g}|_h^2 + |\widetilde{\nabla}\hat{g}|_h^2 + t|\widetilde{\nabla}^2\hat{g}|_h^2 \le C_1 \quad \text{in } N \times [0, T_2]. \tag{3.9}$$

Lemma 3.3. *(cf. Proposition 3.3 of* [CZ]*) Suppose $g(t)$ and $\overline{g}(t)$ are both solutions of (0.1) in $M \times (0, T)$ with $g(0) \equiv \overline{g}(0)$ on M and both $Rm(g(t))$ and $Rm(\overline{g}(t))$ satisfy (0.8) for some constant $k_0 > 0$. Let F and $\overline{F}$ be the solutions of (2.31) in $M \times (0, T_1(g))$ and in $M \times (0, T_1(\overline{g}))$ for some constants $0 < T_1(g), T_1(\overline{g}) \le T$ given by Theorem 2.9 corresponding to Ricci flows $g(t)$ and $\overline{g}(t)$ respectively. Let $T_2(g)$ and $T_2(\overline{g})$ be the constants given by Lemma 3.2. Let $\hat{g}(t) = (F(\cdot,t)^{-1})^*(g(t))$ and $\hat{\overline{g}}(t) = (F(\cdot,t)^{-1})^*(\overline{g}(t))$ be the push forward metric of $g(t)$ and $\overline{g}(t)$ on N by F. Then $\hat{g}(t) \equiv \hat{\overline{g}}(t)$ on $M \times (0, T_0)$ where $T_0 = \min(T_2(g), T_2(\overline{g}))$.*

Proof. A proof of this lemma (Proposition 3.3 of [CZ]) is given in [CZ]. However the proof given in [CZ] is not correct because the deduction of the last two inequalities on P.151 of [CZ] assumed that one can interchange differentiation and taking limit as $\varepsilon \to 0$ which is not true in general. For the sake of completeness we will modify their argument and give a correct proof of the result here. We will use the technique of proof of Theorem 2.6 to proof this lemma. We first recall that by the proof on P.150-151 of [CZ], we have

$$\frac{\partial}{\partial t}|\hat{g} - \hat{\overline{g}}|_h^2 \le \hat{g}^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta|\hat{g} - \hat{\overline{g}}|_h^2 + \frac{C_2}{\sqrt{t}}|\hat{g} - \hat{\overline{g}}|_h^2 \quad \text{in } N \times (0, T_0) \tag{3.10}$$

for some constant $C_2 > 0$ where $h = g(0) = \overline{g}(0)$ and $\widetilde{\nabla}$ is the covariant derivative of h. For any $R_1 \ge 1$ we let

$$u(y,t) = |\hat{g} - \hat{\overline{g}}|_h^2 \widetilde{\phi}(y) \quad \forall y \in N, 0 \le t \le T_0.$$

where $\widetilde{\phi}(y) = \phi(\rho(y)/R_1)$ and $\rho(y) = \rho(p_0, y)$. We first suppose that y is not a cut point of p_0. Then by (3.8), (3.9), (3.10), and Lemma 1.2 for any $0 < t \le T_0$,

$$\begin{aligned}
&u_t - \hat{g}^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta u \\
=&\left[\left(\frac{\partial}{\partial t} - \hat{g}^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\right)|\hat{g} - \hat{\overline{g}}|_h^2\right]\widetilde{\phi} - 2\hat{g}^{\alpha\beta}\widetilde{\nabla}_\alpha|\hat{g} - \hat{\overline{g}}|_h^2 \cdot \widetilde{\nabla}_\beta\widetilde{\phi} - (\hat{g}^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\widetilde{\phi})|\hat{g} - \hat{\overline{g}}|_h^2 \\
\le&\frac{C_2}{\sqrt{t}}|\hat{g} - \hat{\overline{g}}|_h^2\widetilde{\phi} + (4/R_1)|\hat{g}|_h|\hat{g} - \hat{\overline{g}}|_h|\widetilde{\nabla}_\alpha\hat{g} - \widetilde{\nabla}_\alpha\hat{\overline{g}}|_h|\phi'||\widetilde{\nabla}\rho| \\
&\qquad + |\hat{g} - \hat{\overline{g}}|_h^2\left(\frac{|\phi'|\hat{g}^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\rho}{R_1} + \frac{|\phi''||\hat{g}^{\alpha\beta}||\widetilde{\nabla}\rho|^2}{R_1^2}\right) \\
\le&C_2\frac{u}{\sqrt{t}} + \frac{C_3'}{R_1} + \frac{2C_3'}{R_1}|\phi'|\hat{g}^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\rho
\end{aligned} \tag{3.11}$$

for some constant $C_3' > 0$. By the Hessian comparison theorem [SY], P.309-310 of [S2], and (3.8),

$$\begin{aligned}
&\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \rho \le \frac{1+\sqrt{k_0}\rho}{\rho} h_{\alpha\beta} \\
\Rightarrow\quad &\hat{g}^{\alpha\beta}\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \rho \le \hat{g}^{\alpha\beta} h_{\alpha\beta}\frac{1+\sqrt{k_0}\rho}{\rho} \le nC_1 \frac{1+\sqrt{k_0}\rho}{\rho}.
\end{aligned}\tag{3.12}$$

By (3.11) and (3.12),

$$\begin{aligned}
&u_t - \hat{g}^{\alpha\beta}\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta u \le C_2 \frac{u}{\sqrt{t}} + \frac{C_4}{R_1} &&\text{in } (N \setminus \mathrm{Cut}_h(y_0)) \times (0, T_0] \\
\Rightarrow\quad &\left(\frac{\partial}{\partial t} - \hat{g}^{\alpha\beta}\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta\right)(ue^{-2C_2\sqrt{t}}) \le \frac{C_4}{R_1} e^{-2C_2\sqrt{t}} < \frac{2C_4}{R_1} &&\text{in } (N \setminus \mathrm{Cut}_h(y_0)) \times (0, T_0].
\end{aligned}\tag{3.13}$$

for some constant $C_4 > 0$. Let

$$q(y,t) = u(y,t)e^{-2C_2\sqrt{t}} - (2C_4/R_1)t.$$

Then by (3.13),

$$q_t < \hat{g}^{\alpha\beta}\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta q \quad \text{in } (N \setminus \mathrm{Cut}_h(p_0)) \times (0, T_0]. \tag{3.14}$$

By (3.14) and an argument similar to the proof of Theorem 2.6 the function q attains its maximum on $\partial_p(B_h(y_1, \delta) \times (0, T_0))$. Hence

$$\begin{aligned}
&q(y,t) \le \max_{\partial_p(B_h(y_0,R_1)\times[0,T_0])} q(y,t) = 0 &&\text{in } B_h(y_0, R_1) \times [0, T_0] \\
\Rightarrow\quad &|\hat{g} - \hat{\overline{g}}|_h^2 \phi((\rho(p_0, y))/R_1) \le (2C_4/R_1)te^{2C_2\sqrt{t}} &&\text{in } \overline{B_h(p_0, R_1)} \times [0, T_0] \quad \forall R_1 > 1 \\
\Rightarrow\quad &|\hat{g} - \hat{\overline{g}}|_h^2 = 0 &&\text{in } N \times [0, T_0] \text{ as } R_1 \to \infty
\end{aligned}$$

and the lemma follows.

By the same argument as the proof on P.152 of [CZ] but with Lemma 3.4 replacing Proposition 3.3 in the proof there we get the following uniqueness theorem.

Theorem 3.5. *Suppose $g(t)$ and $\overline{g}(t)$ are both solutions of (0.1) in $M \times (0, T)$ with $g(0) \equiv \overline{g}(0)$ on M and both $Rm(g(t))$ and $Rm(\overline{g}(t))$ satisfy (0.8) for some constant $k_0 > 0$. Then $g(t) \equiv \overline{g}(t)$ on $M \times (0, T)$.*

References

[AM]U. Abresch and W.T. Meyer, *Injectivity radius estimates and sphere theorems, P.1–47 in Comparison Geometry, Karsten Grove and P. Petersen editors, Mathematical Sciences Research Institute Publications Vol. 30, Cambridge University Press, 1997.*

[CGT]J. Cheeger, M. Gromov and M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifold*, J. Differential Geometry **17** (1982), 15–53.

[CZ]B.L. Chen and X.P. Zhu, *Uniqueness of the Ricci flow on complete noncompact manifolds*, J. Differential Geometry **74** (2006), 119–154.

[CLY]S.Y. Cheng, P. Li and S.T. Yau, *On the upper estimate of the heat kernel of a complete Riemannian manifold*, Amer. J. Math **103(5)** (1981), 1021–1063.

[Ch]I. Chavel, *Riemannian geometry:A modern introduction*, Cambridge University Press, Cambridge, United Kingdom, 1995.

[CG] J. Cheeger and M. Gromov, *Collapsing Riemannian manifolds while keeping their curvature bounded*, J. Differential Geometry **23** (1996), 309–346.

[CGT]J. Cheeger, M. Gromov and M. Taylor, *Finite propagation speed, kernel estimates for functions of the laplace operator, and the geometry of complete Riemannian manifolds*, J. Differential Geometry **17** (1982), 15–53.

[D] D.M. Deturck, *Deforming metrics in the direction of their Ricci tensors (improved version), in Collected Papers on Ricci Flow, ed. H.D. Cao, B. Chow, S.C. Chu and S.T. Yau, International Press, Somerville, MA, 2003.*

[G] J.D.E. Grant, *Injectivity radius estimates I, http://www.math.univie.ac.at/∼grant/papers/talk.pdf.*

[GW1]R.E. Green and H. Wu, *On the subharmonic and plurisubharmonicity of geodesically convex functions*, Indiana Univ. Math. J. **22(7)** (1973), 641–653.

[GW2]R.E. Green and H. Wu, C^∞ *approximations of convex, subharmonic, and plurisubharmonic functions*, Annales Sci. de l'É.N.S. **12** (1979), 47–84.

[H1]R.S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17(2)** (1982), 255–306.

[H2]R.S. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24(2)** (1986), 153–179.

[H3]R.S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993),7–136, International Press, Cambridge, MA, 1995.

[Hs] S.Y. Hsu, *Removable singularities of semilinear parabolic equations*, Adv. Differential Equations **15(1-2)** (2010), 137–158.

[J] J. Jost, *Riemannian geometry and geometric analysis, 2nd. ed.*, Springer-Verlag, Berlin, Heidelberg, Germany, 1998.

[LSU] O.A. Ladyzenskaya, V.A. Solonnikov, and N.N. Uraltceva, *Linear and quasilinear equations of parabolic type*, Transl. Math. Mono. Vol 23, Amer. Math. Soc., Providence, R.I., 1968.

[LT]P. Lu and G. Tian, *Uniqueness of solutions in the work of Perelman, http://www.math.lsa.umich.edu /∼lott/ricciflow/StanUniqWork2.pdf.*

[SY]R. Schoen and S.T. Yau, *Lectures on differential geometry, in 'Conference proceedings and Lecture Notes in Geometryand Topology', 1*, International Press, 1994.

[S1] W.X. Shi, *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. **30** (1989), 223–301.

[S2] W.X. Shi, *Ricci deformation of the metric on complete non-compact Riemannian manifolds*, J. Differential Geom. **30** (1989), 303–394.

[W1]J.H.C. Whitehead, *Convex regions in the geometry of paths*, Quart. J. Math. **3** (1932), 33–42.

[W2]J.H.C. Whitehead, *Convex regions in the geometry of paths–Addendum*, Quart. J. Math. **4** (1933), 226–27.

E-mail address:syhsu@math.ccu.edu.tw